On the number of minimum dominating sets and total dominating sets in forests

Jan Petr
(joint work with Julien Portier and Leo Versteegen)

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Dominating sets and total dominating sets

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- We look at structure of a forest F with $\Gamma(F) = f(\gamma, s)$ 'near its ends'.

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- Assuming $f(\gamma, s) > \alpha^s \beta^{\gamma-s}$, γ smallest possible, $\Gamma(F) = f(\gamma, s)$.
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- **Claim**: A support vertex $v \in V(F)$ with at most one non-leaf neighbour w is not a strong support vertex.
- Case 1: w is a strong support vertex. $\Gamma(F) \leq f(\gamma - 1, s - 1) \leq \alpha^{s-1}\beta^{\gamma-s} < \alpha^s \beta^{\gamma-s}$.
- Case 2: w is not a strong support vertex. $\Gamma(F) \leq f(\gamma - 1, s - 1) + f(\gamma - 1, s) \leq \alpha^s \beta^{\gamma-s}(\alpha^{-1} + \beta^{-1}) = \alpha^s \beta^{\gamma-s}$.

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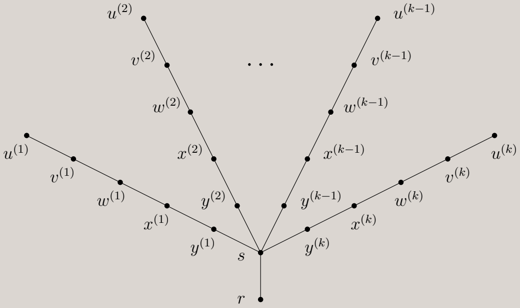
- **Theorem (PPV)**: For every positive integer γ, there exists a tree with domination number γ that has more than $\frac{2}{5}\sqrt{5}^{\gamma}$ minimum dominating sets.



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- **Theorem (R)**: Let $\lambda = \sqrt[13]{95} \approx 1.419$.
  (1) A tree with n vertices has at most $2\lambda^{n-2} < 0.993 \cdot \lambda^n$ minimal dominating sets.
  (2) For every n, there is a tree with at least $0.650 \cdot \lambda^n$ minimal dominating sets.

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unbounded

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- **Conjecture (HMR)**: If a tree T has order n at least 2 and total domination number γ_t, then $\Gamma_t(T) \leq \left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}$.

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- **Theorem (HMR)**: If a forest F has order n, no isolated vertex, and total domination number γ_t, then $\Gamma_t(T) \leq (8\sqrt{e})^{\gamma_t}(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}})$.

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- **Theorem (PPV)**: There exists $c > 1$ and trees T with arbitrarily large total domination number γ_t and order n with $\Gamma_t(T) \geq c^{\gamma_t}\left(\frac{n-\frac{\gamma_t}{2}}{\frac{\gamma_t}{2}}\right)^{\frac{\gamma_t}{2}}$.

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- **Theorem (T)**: If a tree has order n at least 2, then $\Gamma_t(F) \leq \sqrt{2}^n \approx 1.414^n$.

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- Kratsch: There are graphs containing $15^{n/6} \approx 1.570^n$ minimal dominating sets.