Cheeger-type inequalities for the second largest spectral gap from 1 of the normalized Laplacian

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based on joint work with Lies Beers and Raffaella Mulas

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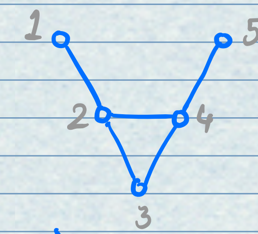
Normalized Laplacian of a graph



$G=(V,E)$ with no isolated vertices, $|V|=n$

adjacency matrix A

degree matrix D

$$A=\begin{pmatrix} 0&1&0&0&0\\ 1&0&1&1&0\\ 0&1&0&1&0\\ 0&1&1&0&1\\ 0&0&0&1&0 \end{pmatrix} \qquad D=\begin{pmatrix} 1&0&0&0&0\\ 0&3&0&0&0\\ 0&0&2&0&0\\ 0&0&0&3&0\\ 0&0&0&0&1 \end{pmatrix}$$

(left) normalized Laplacian $L = Id - D^{-1}A$

eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$

$$L=\begin{pmatrix} 1&-1&0&0&0\\ -1/3&1&-1/3&-1/3&0\\ 0&-1/2&1&-1/2&0\\ 0&-1/3&-1/3&1&-1/3\\ 0&0&0&-1&1 \end{pmatrix}$$

The eigenvalues of L satisfy:

$0 \leq (7-\sqrt{13})/6 \leq 1 \leq 5/3 \leq (7+\sqrt{13})/6$

- multiplicity of 0 is the number of connected components of G (in particular, $\lambda_1 = 0$)
- $\frac{n}{n-1} \leq \lambda_n \leq 2$

equality iff $G=K_n$ equality iff at least one component is bipartite

Two applications of L:

- DL can be used to compute the number of spanning trees of G (Kirchhoff, 1847)
- bounding mixing times of random walks

Rayleigh quotients

$C(V)$ ... (real) vector space of functions $f: V \to \mathbb{R}$

inner product $\langle \cdot, \cdot \rangle : C(V) \times C(V) \to \mathbb{R}$

$$\langle f, g \rangle = \sum_{v \in V} \deg(v) \cdot f(v) \cdot g(v)$$

L is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, i.e. $\langle Lf, g \rangle = \langle f, Lg \rangle$

Rayleigh quotient of $g \in C(V) \setminus \{0\}$: $RQ_L(g) = \dfrac{\langle Lg, g \rangle}{\langle g, g \rangle}$

Thm (Courant-Fischer-Weyl min-max principle):

Let $B \in \mathbb{R}^{V \times V}$ be self-adjoint with respect to $\langle \cdot, \cdot \rangle$,

with eigenvalues $\beta_1 \leq \ldots \leq \beta_n$.

Then, for all $i \in [n]$:

$$\beta_i = \min_{\substack{H \leq C(V) \\ \dim H = i}} \max_{\substack{g \in H \\ g \neq 0}} RQ_L(g) = \max_{\substack{H \leq C(V) \\ \dim H = n-i+1}} \min_{\substack{g \in H \\ g \neq 0}} RQ_L(g).$$

Second largest spectral gap from 1

$\tau = \max_{i \neq 1} |1-\lambda_i| = \max\{1-\lambda_2, \lambda_n - 1\}$

$\tau = 1$ iff G is disconnected or bipartite

Why study τ?

- bounds the rate of convergence to the stationary distribution of the (lazy) simple random walk on G
- Ramanujan graphs: connected d-regular graphs such that

$$\max_{i:\, \lambda_i \neq 0,2} |1-\lambda_i| \leq \frac{2\sqrt{d-1}}{d}$$

(non-bipartite Ramanujan graphs satisfy $\tau \leq \frac{2\sqrt{d-1}}{d}$)

Set $M = (\mathrm{Id} - L)^2 = (D^{-1}A)^2$. Its eigenvalues are $(1-\lambda_1)^2, \ldots, (1-\lambda_n)^2$. $M_{vu} = \sum_{w \in \mathcal{N}(v) \cap \mathcal{N}(u)} \frac{1}{\deg w \cdot \deg v} \geqslant 0$.

$$RQ_M(f) = \frac{1}{\sum_{w \in V} \deg w \cdot f(w)^2} \sum_{w \in V} \frac{1}{\deg w} \cdot \Big(\sum_{v \in \mathcal{N}(w)} f(v)\Big)^2$$

Related work: analysing $\varepsilon = \min_{i \neq 1} |1-\lambda_i|$ (Jost, Mulas, Zhang, 2023)

Cheeger constants and inequalities

For $S \subseteq V$, define $\bar{S} = V \setminus S$

$$\operatorname{vol} S = \sum_{w \in S} \deg w$$

and if $S \neq \emptyset$, $\qquad h(S) = \dfrac{e(S, \bar{S})}{\operatorname{vol} S}$

The Cheeger constant of G is $\displaystyle h = \min_{\substack{\emptyset \neq S \subseteq V \\ \operatorname{vol} S \leq \operatorname{vol} V/2}} h(S)$.

(Pólya, Szegö, 1951)

The dual Cheeger constant of G is $\displaystyle \bar{h} = \max_{\substack{V = S_1 \dot\cup S_2 \dot\cup S_3 \\ S_1, S_2 \neq \emptyset}} \frac{2 \cdot e(S_1, S_2)}{\operatorname{vol}(S_1) + \operatorname{vol}(S_2)}$

(Bauer, Hua, Jost, 2014)

Cheeger inequalities: $1 - \sqrt{1 - h^2} \leq \lambda_2 \leq 2h$

Dual Cheeger inequalities: $2\bar{h} \leq \lambda_n \leq 1 + \sqrt{1 - (1 - \bar{h})^2}$.

Bounding τ using Cheeger and dual Cheeger constants

Set $\hat{h} = \min\{h, 1-\bar{h}\}$. Recall $\tau = \max\{1-\lambda_2, \lambda_n - 1\}$,

so $1-\tau = \min\{\lambda_2, 2-\lambda_n\}$.

Cheeger inequalities: $1-\sqrt{1-h^2} \le \lambda_2 \le 2h$

Dual Cheeger inequalities: $1-\sqrt{1-(1-\bar{h})^2} \le 2-\lambda_n \le 2(1-\bar{h})$

Since $f(x) = 1-\sqrt{1-x^2}$ is increasing on $[0,1]$,

we have $1-\sqrt{1-\hat{h}^2} \le 1-\tau \le 2\hat{h}$.

Here, we use a minimum of two constants.

Could there be a single "natural" constant for bounding τ?

Cheeger-type constant $\tilde{h}$

For $\emptyset \neq S \subseteq V$, define $\tilde{h}(S) = \frac{1}{\operatorname{vol} S} \sum_{w \in V} \frac{e(w,S)^2}{\deg w}$

and set $\tilde{h} = \max\limits_{\substack{\emptyset \neq S \subseteq V \\ \operatorname{vol} S \leq \operatorname{vol} V/2}} \tilde{h}(S)$.

Probabilistic interpretation: $\tilde{h}(S) = \sum_{w \in V} \frac{e(w,S)}{\operatorname{vol} S} \cdot \frac{e(w,S)}{\deg w}$

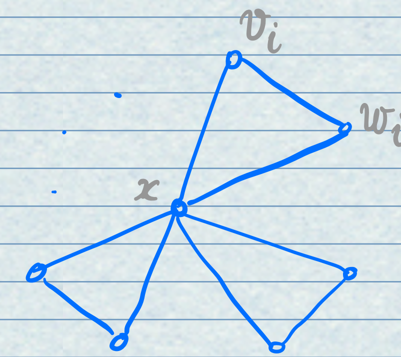
Assume that we select a starting vertex with probability proportional to its degree. Then $\tilde{h}(S)$ is the probability that a random walk starting in S returns to S after two steps.

Compare with $h(S) = \frac{e(S,\bar{S})}{\operatorname{vol} S}$, the probability that a one-step random walk starting in S leaves S.

<u>Proposition</u>: $\tilde{h} \leq 1$, equality iff G is bipartite or disconnected.

Proof idea: $$\tilde{h}(S) = \frac{1}{\sum_{w \in S} \deg w} \cdot \sum_{w \in V} \frac{1}{\deg w} \cdot e(w,S)^2 = \frac{\sum_{w \in V} \frac{1}{\deg w} \left(\sum_{v \in N(w)} \mathbb{1}_S(v) \right)^2}{\sum_{w \in V} \deg w \cdot \mathbb{1}_S(w)^2}$$
$$= RQ_M(\mathbb{1}_S)$$

Example: m-petal graphs



Let $\phi \neq S \subseteq V$ s.t. $\text{vol}\, S \leq \text{vol}\, V/2$.

Let $c = \mathbb{1}\{x \in S\}$,

$$k = \sum_{\substack{v \in V \\ v \neq x}} \mathbb{1}\{v \in S\}.$$

Then $\text{vol}\, S = 2mc + 2k$. Since $\text{vol}\, S \leq \text{vol}\, V/2 = 3m$,

either $c = 0$ & $k \leq \lfloor \frac{3m}{2} \rfloor$, $\Rightarrow$ $\tilde{h}(S) = \frac{k+m}{4m}$

or $c = 1$ & $k \leq \lfloor \frac{m}{2} \rfloor$. $\Rightarrow$ $\tilde{h}(S) = \frac{k+2m}{4m}$

$$\tilde{h}(S) = \frac{1}{\text{vol}\, S} \cdot \sum_{w \in V} \frac{e(w,S)^2}{\deg w} = \frac{1}{2mc+2k} \cdot \left[\frac{k^2}{2m} + k \cdot \frac{(1+c)^2}{2} + (2m-k) \cdot \frac{c^2}{2} \right]$$

In both cases, $\tilde{h}(S)$ is increasing in k, and hence

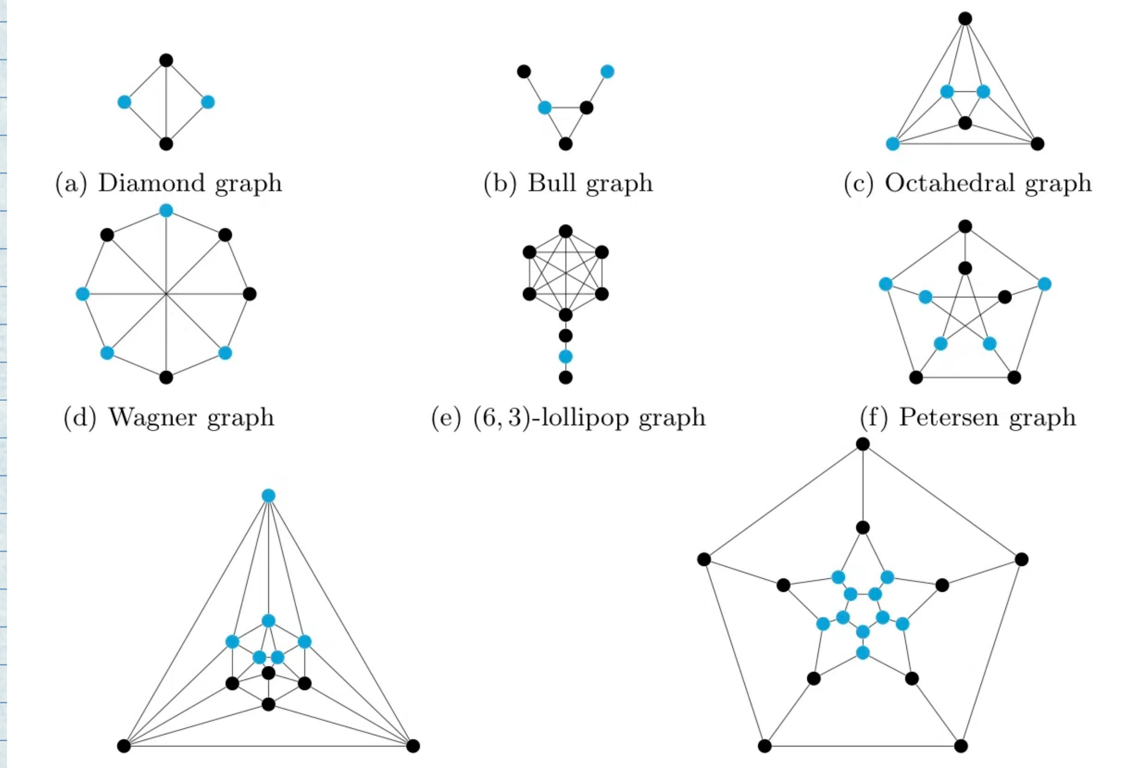
$$\tilde{h} = \begin{cases} \frac{5}{8} & \text{if } m \text{ is even} \\ \frac{5m-1}{8m} & \text{if } m \text{ is odd} \end{cases}$$

More examples

- K_n : $\quad \tilde{h} = \begin{cases} \frac{1}{2}\left(1+\frac{1}{(n-1)^2}\right) & \text{if } n \text{ is even} \\ \frac{1}{2}\left(1-\frac{n-3}{(n-1)^2}\right) & \text{if } n \text{ is odd} \end{cases}$

- C_{2m+1} : $\quad \tilde{h} = \frac{2m-1}{2m}$

Graph	n	$\tilde{h}$
Diamond graph	4	2/3
Bull graph	5	17/24
Octahedral graph	6	7/12
Wagner graph	8	7/9
(6, 3)-lollipop graph	9	3/4
Petersen graph	10	31/45
Icosahedral graph	12	3/5
Dodecahedral graph	20	7/9



(a) Diamond graph (b) Bull graph (c) Octahedral graph (d) Wagner graph (e) (6, 3)-lollipop graph (f) Petersen graph (g) Icosahedral graph (h) Dodecahedral graph

Question: Is there a function f s.t. $\tilde{h} \geq f(n)$ for all graphs on n vertices, satisfying $\lim_{n\to\infty} f(n) = \frac{1}{2}$?

Main theorem – Cheeger-type inequality

Thm (BMP, 26+): $\sqrt{1-(1-\tilde{h})^2} \geq \tau^2 \geq 2\tilde{h}-1$

[Compare with $1-\sqrt{1-h^2} \leq \lambda_2 \leq 2h$

$$2\bar{h} \leq \lambda_n \leq 1+\sqrt{1-(1-\bar{h})^2}$$

$$1-2\hat{h} \leq \tau \leq \sqrt{1-\hat{h}^2}.$$]

Proof of the lower bound $\tau^2 \geq 2\tilde{h} - 1$

Thm (BMP, 26+): $\tau^2 \geq \max\limits_{\emptyset \neq S \subseteq V} \underbrace{\left(\frac{\operatorname{vol} V}{\operatorname{vol} \bar{S}} \cdot \left(\frac{1}{\operatorname{vol} S} \cdot \sum_{w \in V} \frac{e(w,S)^2}{\deg w} \right) - \frac{\operatorname{vol} S}{\operatorname{vol} \bar{S}} \right)}_{(*)}$

Thm $\Rightarrow$ $\tau^2 \geq 2\tilde{h} - 1$:

Let $\emptyset \neq S \subseteq V$ such that $\operatorname{vol} S \leq \operatorname{vol} V/2$.

By Thm,
$$\tau^2 \geq \frac{\operatorname{vol} V}{\operatorname{vol} \bar{S}} \cdot \tilde{h}(S) - \frac{\operatorname{vol} S}{\operatorname{vol} \bar{S}}$$
$$= \tilde{h}(S) + \underbrace{\frac{\operatorname{vol} S}{\operatorname{vol} \bar{S}}}_{\leq 1} \underbrace{(\tilde{h}(S) - 1)}_{\leq 0}$$
$$\geq 2\tilde{h}(S) - 1 \qquad \square$$

Proof sketch of Thm: Fix $\emptyset \neq S \subsetneq V$.

Define $f: V \to \mathbb{R}$ by $f(v) = \begin{cases} 1, & v \in S \\ -\frac{\operatorname{vol} S}{\operatorname{vol} \bar{S}}, & v \in \bar{S} \end{cases}$.

Then $\tau^2 \geq RQ(f) = \ldots = (*)$

Note: for bipartite and disconnected graphs $\tau^2 = 1 = 2\tilde{h} - 1$.

Proof of the upper bound $\tau^2 \le \sqrt{1-(1-\tilde{h})^2}$

Recall $\tau^2 = \max\limits_{\substack{f\in C(V)\setminus\{0\} \\ \sum_{w\in V} \deg w \cdot f(w) = 0}} RQ_M(f) = \max\limits_{\substack{f\in C(V)\setminus\{0\} \\ \sum_{w\in V} \deg w \cdot f(w) = 0}} \dfrac{\sum_{w\in V} \frac{1}{\deg w}\left(\sum_{v\in N(w)} f(v)\right)^2}{\sum_{w\in V} \deg w \cdot f(w)^2}$

Step 1: Find $f\in C(V)\setminus\{0\}$ s.t.

- $\tau^2 \le RQ_M(f)$
- $\operatorname{vol} \operatorname{supp}(f) \in \left(0, \frac{\operatorname{vol} V}{2}\right]$
- $1 \in \operatorname{Range}(f) \subset [0,1]$

useful for a probabilistic argument

Step 2: Find $\emptyset \ne S \subset V$ satisfying

- $RQ_M(f) \le \sqrt{1-(1-\tilde{h}(S))^2}$
- $\operatorname{vol} S \le \operatorname{vol} V/2$

Proof of the upper bound $\tau^2 \le \sqrt{1-(1-\tilde{h})^2}$

Step 1: Find $f \in C(V)\setminus\{0\}$ s.t.
- $\tau^2 \le RQ_M(f)$
- $\mathrm{vol}\ \mathrm{supp}(f) \in (0, \frac{\mathrm{vol}\ V}{2}]$
- $1 \in \mathrm{Range}(f) \subset [0,1]$

Step 2: Find $\emptyset \ne S \subset V$ satisfying
- $RQ_M(f) \le \sqrt{1-(1-\tilde{h}(S))^2}$
- $\mathrm{vol}\ S \le \mathrm{vol}\ V/2$

Let $g \in C(V)\setminus\{0\}$ satisfy

$$\sum_{w\in V} \deg w \cdot g(w) = 0 \quad \text{and} \quad \tau^2 = RQ_M(g)$$

$\Rightarrow$ g contains at least one positive and one negative value

For any $c \in \mathbb{R}$ we have $RQ_M(g - c\cdot 1) \ge RQ_M(g)$

Pick c such that $\mathrm{vol}\{v : g(v) - c \ge 0\} \ge \frac{\mathrm{vol}\ V}{2}$

and $\mathrm{vol}\{v : g(v) - c \le 0\} \ge \frac{\mathrm{vol}\ V}{2}$

In particular, $\mathrm{vol}\{v : g(v) - c < 0\} \le \frac{\mathrm{vol}\ V}{2}$

and $\mathrm{vol}\{v : g(v) - c > 0\} \le \frac{\mathrm{vol}\ V}{2}$

Set $\tilde{g} = g - c\cdot 1$, $\tilde{g}^+ = \max\{\tilde{g}, 0\}$, $\tilde{g}^- = \max\{-\tilde{g}, 0\}$, so $\tilde{g} = \tilde{g}^+ - \tilde{g}^-$

$$\mathrm{vol}\ \mathrm{supp}(\tilde{g}^+) \le \frac{\mathrm{vol}\ V}{2}, \quad \mathrm{vol}\ \mathrm{supp}(\tilde{g}^-) \le \frac{\mathrm{vol}\ V}{2}$$

Proof of the upper bound $\tau^2 \le \sqrt{1-(1-\tilde{h})^2}$

Step 1: Find $f \in C(V) \setminus \{0\}$ s.t.

- $\tau^2 \le RQ_M(f)$
- $\mathrm{vol}\ \mathrm{supp}(f) \in (0, \frac{\mathrm{vol}\ V}{2}]$
- $1 \in \mathrm{Range}(f) \subset [0,1]$

Step 2: Find $\phi \ne S \subset V$ satisfying

- $RQ_M(f) \le \sqrt{1-(1-\tilde{h}(S))^2}$
- $\mathrm{vol}\ S \le \mathrm{vol}\ V/2$

Claim: At least one of $\tilde{f} \in \{\tilde{g}^+, \tilde{g}^-\}$ satisfies $\tilde{f} \ne 0$ and $RQ_M(\tilde{g}) \le RQ_M(\tilde{f})$.

Pf of Claim: At least one of $\{\tilde{g}^+, \tilde{g}^-\}$ is non-zero.

If $\tilde{g}^+ = 0$, set $\tilde{f} = \tilde{g}^-$ (and vice versa)

If both are non-zero:

$$RQ_M(\tilde{g}) = \frac{\langle \tilde{g}^+ - \tilde{g}^-, M(\tilde{g}^+ - \tilde{g}^-)\rangle}{\langle \tilde{g}^+ - \tilde{g}^-, \tilde{g}^+ - \tilde{g}^-\rangle}$$

$$= \frac{\langle \tilde{g}^+, M\tilde{g}^+\rangle - \langle \tilde{g}^+, M\tilde{g}^-\rangle - \langle \tilde{g}^-, M\tilde{g}^+\rangle + \langle \tilde{g}^-, M\tilde{g}^-\rangle}{\langle \tilde{g}^+, \tilde{g}^+\rangle + \langle \tilde{g}^-, \tilde{g}^-\rangle}$$

$$\le \frac{\langle \tilde{g}^+, M\tilde{g}^+\rangle + \langle \tilde{g}^-, M\tilde{g}^-\rangle}{\langle \tilde{g}^+, \tilde{g}^+\rangle + \langle \tilde{g}^-, \tilde{g}^-\rangle}$$

$$\le \max\{RQ_M(\tilde{g}^+), RQ_M(\tilde{g}^-)\} \quad \square$$

Finally, set $f = \frac{1}{\max\{\tilde{f}(v) : v \in V\}} \cdot \tilde{f}$, concluding Step 1.

Proof of the upper bound $\tau^2 \le \sqrt{1-(1-\tilde{h})^2}$

Step 1: Find $f \in C(V) \setminus \{0\}$ s.t.
- $\tau^2 \le RQ_M(f)$
- $\mathrm{vol}\ \mathrm{supp}(f) \in (0, \frac{\mathrm{vol}\ V}{2}]$
- $1 \in \mathrm{Range}(f) \subset [0,1]$

Step 2: Find $\phi \neq S \subset V$ satisfying
- $RQ_M(f) \le \sqrt{1-(1-\tilde{h}(S))^2}$
- $\mathrm{vol}\ S \le \mathrm{vol}\ V/2$

Consider the random set $S_t = \{v : f(v) \ge t\}$, where $t^2 \sim U[0,1]$.

Then:
- $\mathbb{P}[v \in S_t] = \mathbb{P}[f(v)^2 \ge t^2] = f(v)^2$
- $\mathbb{P}[u, v \in S_t] = \min\{f(u)^2, f(v)^2\}$
- $\mathbb{P}[u \in S_t \vee v \in S_t] = \max\{f(u)^2, f(v)^2\}$
- $\mathbb{P}[S_t = \phi] = 0$ (since $1 \in \mathrm{Range}(f)$)
- $S_t \subset \mathrm{supp}(f)$ almost surely ($\mathbb{P}[t=0]=0$)

  $\Rightarrow \mathrm{vol}\ S_t \in (0, \mathrm{vol}\ V/2)$ almost surely
- $\mathbb{E}[\mathrm{vol}\ S_t] = \sum_{w \in V} \deg w \cdot \mathbb{E}[\mathbb{1}_{S_t}(w)] = \sum_{w \in V} \deg w \cdot f(w)^2$

$$RQ_M(f) = \frac{1}{\sum_{w \in V} \deg w \cdot f(w)^2} \sum_{w \in V} \frac{1}{\deg w} \cdot \Big(\sum_{v \in \mathcal{N}(w)} f(v)\Big)^2$$

$$= \frac{1}{\sum_{w \in V} \deg w \cdot f(w)^2} \sum_{w \in V} \frac{1}{\deg w} \sum_{(u,v) \in \mathcal{N}(w)^2} f(u) \cdot f(v)$$

Proof of the upper bound $\tau^2 \le \sqrt{1-(1-\tilde{h})^2}$

Step 1: Find $f \in C(V)\setminus\{0\}$ s.t.

- $\tau^2 \le RQ_M(f)$
- $\mathrm{vol}\ \mathrm{supp}(f) \in \left(0, \frac{\mathrm{vol}\ V}{2}\right]$
- $1 \in \mathrm{Range}(f) \subset [0,1]$

Step 2: Find $\emptyset \ne S \subset V$ satisfying

- $RQ_M(f) \le \sqrt{1-(1-\tilde{h}(S))^2}$
- $\mathrm{vol}\ S \le \mathrm{vol}\ V/2$

$$RQ_M(f) = \frac{1}{\sum_{w\in V} \deg w \cdot f(w)^2} \sum_{w \in V} \frac{1}{\deg w} \cdot \Big(\sum_{v \in \mathcal{N}(w)} f(v)\Big)^2$$

$$= \frac{1}{\sum_{w\in V} \deg w \cdot f(w)^2} \sum_{w \in V} \frac{1}{\deg w} \sum_{(u,v)\in \mathcal{N}(w)^2} \underbrace{f(u)f(v)}_{\min\{f(u),f(v)\}\cdot\max\{f(u),f(v)\}}$$

C-S

$$\le \frac{1}{\sum_{w\in V} \deg w \cdot f(w)^2} \Bigg(\sum_{\substack{w\in V\\(u,v)\in\mathcal{N}(w)^2}} \frac{\min^2}{\deg w}\Bigg)^{1/2} \Bigg(\sum_{\substack{w\in V\\(u,v)\in\mathcal{N}(w)^2}} \frac{\max^2}{\deg w}\Bigg)^{1/2}$$

$$= \frac{1}{\mathbb{E}[\mathrm{vol}\ S_t]} \Bigg(\sum \frac{\mathbb{P}[u,v\in S_t]}{\deg w}\Bigg)^{1/2} \cdot \Bigg(\sum \frac{\mathbb{P}[u\in S_t \vee v \in S_t]}{\deg w}\Bigg)^{1/2}$$

$$= \frac{1}{\mathbb{E}[\mathrm{vol}\ S_t]} \Bigg(\mathbb{E} \sum \frac{\mathbb{1}\{u,v\in S_t\}}{\deg w}\Bigg)^{1/2} \Bigg(\mathbb{E} \sum \frac{\mathbb{1}\{u\in S_t \vee v \in S_t\}}{\deg w}\Bigg)^{1/2}$$

Proof of the upper bound $\tau^2 \le \sqrt{1-(1-\tilde{h})^2}$

Step 1: Find $f \in C(V) \setminus \{0\}$ s.t.
- $\tau^2 \le RQ_M(f)$
- $\mathrm{vol}\ \mathrm{supp}(f) \in (0, \frac{\mathrm{vol}\ V}{2}]$
- $1 \in \mathrm{Range}(f) \subset [0,1]$

Step 2: Find $\emptyset \ne S \subset V$ satisfying
- $RQ_M(f) \le \sqrt{1-(1-\tilde{h}(S))^2}$
- $\mathrm{vol}\ S \le \mathrm{vol}\ V/2$

Note: $\displaystyle\sum_{(u,v)\in \mathcal{N}(w)^2} \mathbb{1}\{u,v \in S_t\} = e(w,S_t)^2$

$$\sum_{(u,v)\in \mathcal{N}(w)^2} \mathbb{1}\{u\in S_t \vee v \in S_t\} = \sum_{(u,v)\in \mathcal{N}(w)^2} \mathbb{1}\{u,v\in S_t\} + \sum_{(u,v)\in \mathcal{N}(w)^2} \mathbb{1}\{u\in S_t \dot\vee v \in S_t\} =$$

$$\frac{e(w,S_t)^2}{\deg w} + 2e(w,S_t)\,e(w,\bar{S}_t)$$

$$RQ_M(f) \le \frac{1}{\mathbb{E}[\mathrm{vol}\ S_t]}\left(\mathbb{E}\sum_{\substack{w\in V\\(u,v)\in \mathcal{N}(w)^2}} \frac{\mathbb{1}\{u,v\in S_t\}}{\deg w}\right)^{1/2}\left(\mathbb{E}\sum_{\substack{w\in V\\(u,v)\in \mathcal{N}(w)^2}} \frac{\mathbb{1}\{u\in S_t \vee v\in S_t\}}{\deg w}\right)^{1/2}$$

$$= \frac{1}{\mathbb{E}[\mathrm{vol}\ S_t]}\cdot\left(\mathbb{E}\Big[\sum_{w\in V}\frac{e(w,S_t)^2}{\deg w}\Big]\right)^{1/2}\cdot\left(\mathbb{E}\Big[\sum_{w\in V}\frac{e(w,S_t)^2}{\deg w} + \frac{2e(w,S_t)\cdot e(w,\bar{S}_t)}{\deg w}\Big]\right)$$

$$= \dots$$

$$= \frac{\mathbb{E}[X_t]^{1/2}\cdot\left(-\mathbb{E}[X_t]+2\,\mathbb{E}[Y_t]\right)^{1/2}}{\mathbb{E}[Y_t]}$$

$$= \sqrt{1-\left(1-\frac{\mathbb{E}[X_t]}{\mathbb{E}[Y_t]}\right)^2}$$

$$X_t := \sum_{w\in V}\frac{e(w,S_t)^2}{\deg w}$$

$$Y_t := \mathrm{vol}\ S_t$$

(note: $\mathbb{P}[Y_t > 0] = 1$)

Proof of the upper bound $\tau^2 \le \sqrt{1-(1-\tilde{h})^2}$

Step 1: Find $f \in C(V) \setminus \{0\}$ s.t.
- $\tau^2 \le RQ_M(f)$
- $\mathrm{vol}\ \mathrm{supp}(f) \in (0, \frac{\mathrm{vol}\ V}{2}]$
- $1 \in \mathrm{Range}(f) \subset [0,1]$

Step 2: Find $\emptyset \neq S \subset V$ satisfying
- $RQ_M(f) \le \sqrt{1-(1-\tilde{h}(S))^2}$
- $\mathrm{vol}\ S \le \mathrm{vol}\ V/2$

$$RQ_M(f) \le \sqrt{1-\left(1-\frac{\mathbb{E}[X_t]}{\mathbb{E}[Y_t]}\right)^2}$$

<u>Lemma</u>: Let X and Y be random variables such that $\mathbb{P}[Y>0]=1$. Then $\mathbb{P}\left[\frac{X}{Y} \ge \frac{\mathbb{E}X}{\mathbb{E}Y}\right] > 0$.

$$RQ_M(f) \le \sqrt{1-\left(1-\frac{X_{t_0}}{Y_{t_0}}\right)^2} \quad \text{for some } t_0.$$

Using that $f(x) = \sqrt{1-(1-x)^2}$ is increasing on $[0,1]$:

$$RQ_M(f) \le \sqrt{1-(1-\tilde{h}(S_0))^2}$$

Concluding step 2 and the whole proof. $\square$

$$X_t := \sum_{w \in V} \frac{e(w,S_t)^2}{\deg w}$$

$$Y_t := \mathrm{vol}\ S_t$$

(note: $\mathbb{P}[Y_t > 0] = 1$)

Proof of the upper bound $\tau^2 \leq \sqrt{1-(1-\tilde{h})^2}$

Equality case: G is disconnected or bipartite:

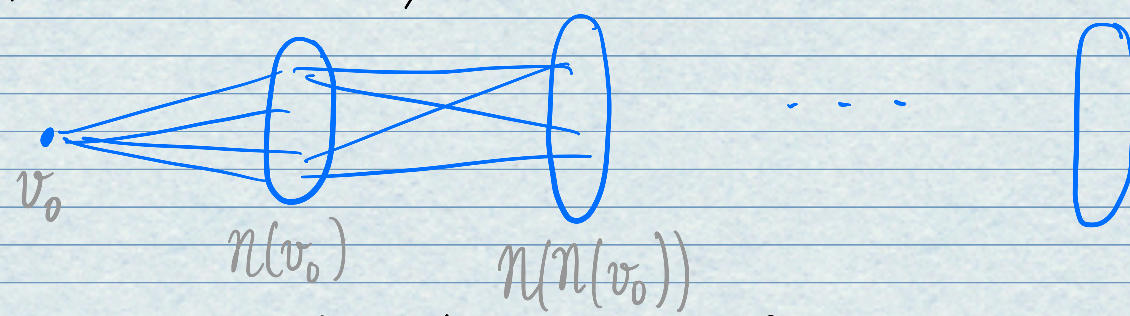
Needed $\displaystyle\sum_{\substack{w \in V \\ (u,v) \in \mathcal{N}(w)^2}} \frac{\min\{f(u), f(v)\} \cdot \max\{f(u), f(v)\}}{\deg w} =$ (Cauchy–Schwarz)

$$\left(\sum_{\substack{w \in V \\ (u,v) \in \mathcal{N}(w)^2}} \frac{\min^2}{\deg w} \right)^{1/2} \cdot \left(\sum_{\substack{w \in V \\ (u,v) \in \mathcal{N}(w)^2}} \frac{\max^2}{\deg w} \right)^{1/2}$$

This implies $f(u) = f(v)$ for all $w \in V$ and all $(u,v) \in \mathcal{N}(w)^2$.

If G is disconnected, then $\tilde{h} = 1$, $\tau = 1$.

If G is connected:



S_0 (S_1)… vertices at an even (odd) distance from v_0

f is constant on S_0 and S_1, and attains at least two values by construction $\Rightarrow$ S_0 and S_1 are independent sets

$\Rightarrow$ G is bipartite. $\square$

Proof of the upper bound $\tau^2 \leq \sqrt{1-(1-\tilde{h})^2}$

Remark: By applying Bernoulli's inequality, one gets

$$\tau^2 \leq 1 - \frac{(1-\tilde{h})^2}{2}$$

with equality iff G is disconnected or bipartite

THANK YOU FOR YOUR ATTENTION!